

# A REMARK ON UNIQUE CONTINUATION FOR THE CAUCHY-RIEMANN OPERATOR

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**ABSTRACT.** In this note we obtain a unique continuation result for the differential inequality  $|\bar{\partial}u| \leq |Vu|$ , where  $\bar{\partial} = (i\partial_y + \partial_x)/2$  denotes the Cauchy-Riemann operator and  $V(x, y)$  is a function in  $L^2(\mathbb{R}^2)$ .

## 1. INTRODUCTION

The unique continuation property is one of the most interesting properties of holomorphic functions  $f \in H(\mathbb{C})$ . This property says that if  $f$  vanishes in a non-empty open subset of  $\mathbb{C}$  then it must be identically zero. Note that  $u \in C^1(\mathbb{R}^2)$  satisfies the Cauchy-Riemann equation  $(i\partial_y + \partial_x)u = 0$  if and only if it defines a holomorphic function  $f(x + iy) \equiv u(x, y)$  on  $\mathbb{C}$ . From this point of view, one can see that a  $C^1$  function satisfying the equation has the unique continuation property.

In this note we consider a class of non-holomorphic functions  $u$  which satisfy the differential inequality

$$|\bar{\partial}u| \leq |Vu|, \quad (1.1)$$

where  $\bar{\partial} = (i\partial_y + \partial_x)/2$  denotes the Cauchy-Riemann operator and  $V(x, y)$  is a function on  $\mathbb{R}^2$ .

The best positive result for (1.1) is due to Wolff [9] (see Theorem 4 there) who proved the property for  $V \in L^p$  with  $p > 2$ . On the other hand, there is a counterexample [8] to unique continuation for (1.1) with  $V \in L^p$  for  $p < 2$ . The remaining case  $p = 2$  seems to be unknown for the differential inequality (1.1), and note that  $L^2$  is a scale-invariant space of  $V$  for the equation  $\bar{\partial}u = Vu$ . Here we shall handle this problem. Our unique continuation result is the following theorem which is based on bounds for a Fourier multiplier from  $L^p$  to  $L^q$ .

**Theorem 1.1.** *Let  $1 < p < 2 < q < \infty$  and  $1/p - 1/q = 1/2$ . Assume that  $u \in L^p \cap L^q$  satisfies the inequality (1.1) with  $V \in L^2$  and vanishes in a non-empty open subset of  $\mathbb{R}^2$ . Then it must be identically zero.*

The unique continuation property also holds for harmonic functions, which satisfy the Laplace equation  $\Delta u = 0$ , since they are real parts of holomorphic functions. This was first extended by Carleman [1] to a class of non-harmonic functions satisfying the inequality  $|\Delta u| \leq |Vu|$  with  $V \in L^\infty(\mathbb{R}^2)$ . There is an extensive literature on later

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developments in this subject. In particular, the problem of finding all the possible  $L^p$  functions  $V$ , for which  $|\Delta u| \leq |Vu|$  has the unique continuation, is completely solved (see [3, 5, 7]). See also the survey papers of Kenig [4] and Wolff [10] for more details, and the recent paper of Kenig and Wang [6] for a stronger result which gives a quantitative form of the unique continuation.

Throughout the paper, the letter  $C$  stands for positive constants possibly different at each occurrence. Also, the notations  $\widehat{f}$  and  $\mathcal{F}^{-1}(f)$  denote the Fourier and the inverse Fourier transforms of  $f$ , respectively.

## 2. A PRELIMINARY LEMMA

The standard method to study the unique continuation property is to obtain a suitable Carleman inequality for relevant differential operator. This method originated from Carleman's classical work [1] for elliptic operators. In our case we need to obtain the following inequality for the Cauchy-Riemann operator  $\bar{\partial} = (i\partial_y + \partial_x)/2$ , which will be used in the next section for the proof of Theorem 1.1:

**Lemma 2.1.** *Let  $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ . For all  $t > 0$ , we have*

$$\| |z|^{-t} f \|_{L^q} \leq C \| |z|^{-t} \bar{\partial} f \|_{L^p} \quad (2.1)$$

if  $1 < p < 2 < q < \infty$  and  $1/p - 1/q = 1/2$ . Here,  $z = x + iy \in \mathbb{C}$  and  $C$  is a constant independent of  $t$ .

*Proof.* First we note that

$$\bar{\partial}(z^{-t} f) = z^{-t} \bar{\partial} f + f \bar{\partial}(z^{-t}) = z^{-t} \bar{\partial} f$$

for  $z \in \mathbb{C} \setminus \{0\}$ . Then the inequality (2.1) is equivalent to

$$\| z^{-t} f \|_{L^q} \leq C \| \bar{\partial}(z^{-t} f) \|_{L^p}.$$

By setting  $g = z^{-t} f$ , we are reduced to showing that

$$\| g \|_{L^q} \leq C \| (i\partial_y + \partial_x) g \|_{L^p}$$

for  $g \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ . To show this, let us first set

$$(i\partial_y + \partial_x) g = h, \quad (2.2)$$

and let  $\psi_\delta : \mathbb{R}^2 \rightarrow [0, 1]$  be a smooth function such that  $\psi_\delta = 0$  in the ball  $B(0, \delta)$  and  $\psi_\delta = 1$  in  $\mathbb{R}^2 \setminus B(0, 2\delta)$ . Then, using the Fourier transform in (2.2), we see that

$$(-\eta + i\xi) \widehat{g}(\xi, \eta) = \widehat{h}(\xi, \eta).$$

Thus, by Fatou's lemma we are finally reduced to showing the following uniform boundedness for a multiplier operator having the multiplier  $m(\xi, \eta) = \psi_\delta(\xi, \eta)/(-\eta + i\xi)$ :

$$\left\| \mathcal{F}^{-1} \left( \frac{\psi_\delta(\xi, \eta) \widehat{h}(\xi, \eta)}{-\eta + i\xi} \right) \right\|_{L^q} \leq C \| h \|_{L^p} \quad (2.3)$$

uniformly in  $\delta > 0$ .

From now on, we will show (2.3) using Young's inequality for convolutions and Littlewood-Paley theorem ([2]). Let us first set for  $k \in \mathbb{Z}$

$$\widehat{Th}(\xi, \eta) = m(\xi, \eta)\widehat{h}(\xi, \eta) \quad \text{and} \quad \widehat{T_k h}(\xi, \eta) = m(\xi, \eta)\chi_k(\xi, \eta)\widehat{h}(\xi, \eta),$$

where  $\chi_k(\cdot) = \chi(2^k \cdot)$  for  $\chi \in C_0^\infty(\mathbb{R}^2)$  which is such that  $\chi(\xi, \eta) = 1$  if  $|(\xi, \eta)| \sim 1$ , and zero otherwise. Also,  $\sum_k \chi_k = 1$ . Now we claim that

$$\|T_k h\|_{L^q} \leq C\|h\|_{L^p} \quad (2.4)$$

uniformly in  $k \in \mathbb{Z}$ . Then, since  $1 < p < 2 < q < \infty$ , by the Littlewood-Paley theorem together with Minkowski's inequality, we get the desired inequality (2.3) as follows:

$$\begin{aligned} \left\| \sum_k T_k h \right\|_{L^q} &\leq C \left\| \left( \sum_k |T_k h|^2 \right)^{1/2} \right\|_{L^q} \\ &\leq C \left( \sum_k \|T_k h\|_{L^q}^2 \right)^{1/2} \\ &\leq C \left( \sum_k \|h_k\|_{L^p}^2 \right)^{1/2} \\ &\leq C \left\| \left( \sum_k |h_k|^2 \right)^{1/2} \right\|_{L^p} \\ &\leq C \left\| \sum_k h_k \right\|_{L^p}, \end{aligned}$$

where  $h_k$  is given by  $\widehat{h_k}(\xi, \eta) = \chi_k(\xi, \eta)\widehat{h}(\xi, \eta)$ . Now it remains to show the claim (2.4). But, this follows easily from Young's inequality. Indeed, note that

$$T_k h = \mathcal{F}^{-1} \left( \frac{\psi_\delta(\xi, \eta)\chi_k(\xi, \eta)}{-\eta + i\xi} \right) * h$$

and by Plancherel's theorem

$$\begin{aligned} \left\| \mathcal{F}^{-1} \left( \frac{\psi_\delta(\xi, \eta)\chi_k(\xi, \eta)}{-\eta + i\xi} \right) \right\|_{L^2} &= \left\| \frac{\psi_\delta(\xi, \eta)\chi_k(\xi, \eta)}{-\eta + i\xi} \right\|_{L^2} \\ &\leq C \left( \int_{|(\xi, \eta)| \sim 2^{-k}} \frac{1}{\eta^2 + \xi^2} d\xi d\eta \right)^{1/2} \\ &\leq C. \end{aligned}$$

Since we are assuming the gap condition  $1/p - 1/q = 1/2$ , by Young's inequality for convolutions, this readily implies that

$$\|T_k h\|_{L^q} \leq \left\| \mathcal{F}^{-1} \left( \frac{\psi_\delta(\xi, \eta)\chi_k(\xi, \eta)}{-\eta + i\xi} \right) \right\|_{L^2} \|h\|_{L^p} \leq C\|h\|_{L^p}$$

as desired.  $\square$

## 3. PROOF OF THEOREM 1.1

The proof is standard once one has the Carleman inequality (2.1) in Lemma 2.1.

Without loss of generality, we may show that  $u$  must vanish identically if it vanishes in a sufficiently small neighborhood of zero. Then, since we are assuming that  $u \in L^p \cap L^q$  vanishes near zero, from (2.1) with a standard limiting argument involving a  $C_0^\infty$  approximate identity, it follows that

$$\| |z|^{-t} u \|_{L^q} \leq C \| |z|^{-t} \bar{\partial} u \|_{L^p}.$$

Thus by (1.1) we see that

$$\begin{aligned} \| |z|^{-t} u \|_{L^q(B(0,r))} &\leq C \| |z|^{-t} V u \|_{L^p(B(0,r))} \\ &\quad + C \| |z|^{-t} \bar{\partial} u \|_{L^p(\mathbb{R}^2 \setminus B(0,r))}, \end{aligned}$$

where  $B(0,r)$  is the ball of radius  $r > 0$  centered at 0. Then, using Hölder's inequality with  $1/p - 1/q = 1/2$ , the first term on the right-hand side in the above can be absorbed into the left-hand side as follows:

$$\begin{aligned} C \| |z|^{-t} V u \|_{L^p(B(0,r))} &\leq C \| V \|_{L^2(B(0,r))} \| |z|^{-t} u \|_{L^q(B(0,r))} \\ &\leq \frac{1}{2} \| |z|^{-t} u \|_{L^q(B(0,r))} \end{aligned}$$

if we choose  $r$  small enough. Here,  $\| |z|^{-t} u \|_{L^q(B(0,r))}$  is finite since  $u \in L^q$  vanishes near zero. Hence we get

$$\begin{aligned} \| (r/|z|)^t u \|_{L^q(B(0,r))} &\leq 2C \| \bar{\partial} u \|_{L^p(\mathbb{R}^2 \setminus B(0,r))} \\ &\leq 2C \| V \|_{L^2} \| u \|_{L^q} \\ &< \infty. \end{aligned}$$

By letting  $t \rightarrow \infty$ , we now conclude that  $u = 0$  on  $B(0,r)$ . This implies  $u \equiv 0$  by a standard connectedness argument.

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